Perturbed Nonlinear Models from Noncommutativity

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By means of the Ehrenfest's Theorem inside the context of a noncommutative Quantum Mechanics it is obtained the Newton's Second Law in noncommutative space. Considering discrete systems with infinite degrees of freedom whose dynamical evolutions are governed by the noncommutative Newton's Second Law we have constructed some alternative noncommutative generalizations of two-dimensional field theories.

PACS numbers: 11.10 Nx, 11.10.Lm

I. INTRODUCTION

In the last years the research work on noncommutative (nc) space-times have been quite extensive. Particularly, the investigations in Noncommutative Field Theories (NCFT) have been very active since the appearance of these theories as low-energy limits of string theories in the presence of magnetic fields [1].

Besides the theoretical research in NCFT, the formulation of Quantum Mechanics on these nc spaces(NCQM) have been considered by different authors [2, 3, 4, 5]. It is worthy to mention that in most of the cases treated it is considered that the time coordinate commutes with the space coordinates. At the Quantum Field Theory level this is the usual procedure also since the theories constructed assuming a nc time coordinate could suffer of unitary and causality problems [6]. On the nc two-dimensional field theories scenario the introduction of a nc time-coordinate is compulsory. In these cases the nc extensions are usually achieved introducing the Moyal product on the original action or as in the case of integrable models starting from a generalization of the zero curvature condition [10] [25]. In order to avoid the non-unitarity in the two-dimensional case, Euclidean models can be considered.

Obviously the existence of a noncommutative space-time at quantum level could have consequences at the classical level too. The classical limit of NCQM with commuting-time has been investigated so far by different authors [5, 7, 8, 9]. In [7, 8, 9] the classical limit is taken at the level of the commutators what lead to "deformed" Poisson brackets. Thus, the time evolution of the coordinates is governed by a deformed Newtonian mechanics. On the other hand in [5] the evolution of the expectation value of the coordinates was investigated using the Ehrenfest's theorem and the classical limit was shown to be non-Newtonian as well. Nevertheless, in that paper, the time evolution of the momenta was not analyzed. Additionally, using this deformed Newtonian Mechanics, some single particle traditional models with a finite number of degrees of freedom have been studied, such as the three-dimensional harmonic oscillator and the Kepler problem [8, 9, 13].

Hence, in this letter we will initially investigate in more detail the Ehrenfest's Theorem inside the context of the NCQM with commuting-time for the coordinates as well as for the momenta. Secondly, taking as a point of departure this deformed Newtonian Mechanics, we will study some mechanical models related to two-dimensional field theories. For this purpose we will follow the traditional approach. That is to approximate the continuous system by a corresponding one containing discrete particles that obey the deformed Newton's Second Law and then apply the continuous limit. In this way it is possible to obtain alternative noncommutative extensions of known two-dimensional field theories where time commutes.

II. THE NEWTON'S SECOND LAW ON NONCOMMUTATIVE SPACE FROM THE EHRENFEST'S THEOREM

An alternative way of investigating the classical limit of a quantum theory is provided by the Ehrenfest's Theorem which states that quantum expectation values of operators follow a classical equation of motion. As we will see the classical equation of motion followed by the expectation values of the operators corresponding to NCQM is not Newtonian and the correction term coincides with the one obtained in [7, 8, 9].

Let us consider a system with n degrees of freedom inside the context of Noncommutative Quantum Mechanics with time a commuting coordinate. This system it is describable for canonical coordinates $(\hat{x}_i, \hat{p}_i)$ which obey the commutation relations,

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \qquad (1)$$

with θ_{ij} an antisymmetric tensor with dimensions of $length^2$. We will consider such systems where the Hilbert

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space is the same of the corresponding commutative system [2]. In this sense the dynamical evolution of the state $|\Psi>$ is given by the usual Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi>=H|\Psi>, \tag{2}$$

but the Hamiltonian will depend on the nc coordinates,

$$H=\frac{\hat{p}\cdot\hat{p}}{2m}+V(\hat{x}). \tag{3}$$

It is possible to define a new coordinate system

$$\tilde{x}_i=\hat{x}_i+\frac{1}{2\hbar}\theta_{ij}\hat{p}_j, \quad \text{and} \quad \tilde{p}_i=\hat{p}_i, \tag{4}$$

where the new coordinates satisfy the usual canonical commutation relations,

$$[\tilde{x}_i,\tilde{x}_j]=0, \quad [\tilde{x}_i,\tilde{p}_j]=i\hbar\delta_{ij}, \quad [\tilde{p}_i,\tilde{p}_j]=0. \tag{5}$$

The Hamiltonian in these new coordinates will read

$$H=\frac{\tilde{p}_i^2}{2m}+V(\tilde{x}_i-\frac{1}{2\hbar}\theta_{ij}\tilde{p}_j). \tag{6}$$

If we consider that the noncommutative parameter is small, the potential can be expanded as

$$V=V(\tilde{x}_i)-\frac{1}{2\hbar}\theta_{ij}\tilde{p}_j\frac{\partial V}{\partial\tilde{x}_i}+O(\theta^2). \tag{7}$$

Now we would like to investigate how the Ehrenfest's theorem of Quantum Mechanics would look in this context. For this reason let us consider that we can still use the usual definition of mean value, because we can consider that the noncommutativity enters as a perturbation. Since the mean value of the time derivative of any operator $\mathcal{O}$ satisfies the relation,

$$<\frac{d\mathcal{O}}{dt}>=\frac{i}{\hbar}<[H,\mathcal{O}]>, \tag{8}$$

it is obtained,

$$\begin{aligned}\frac{d}{dt}<\tilde{x}_i> &= \frac{<\tilde{p}_i>}{m}-\frac{\theta_{ji}}{2\hbar}<\frac{\partial V}{\partial\tilde{x}_j}>,\\ \frac{d}{dt}<\tilde{p}_i> &= <-\frac{\partial V}{\partial\tilde{x}_i}>+\frac{\theta_{kj}}{2\hbar}<\tilde{p}_j\frac{\partial^2 V}{\partial\tilde{x}_k\partial\tilde{x}_i}>,\end{aligned} \tag{9}$$

which are the relations corresponding to the Ehrenfest's theorem up to first order in θ and they show us that the mean values of the coordinates and momentum should follow classical equations inside the context of a deformed Newtonian mechanics. In fact the classical limit will exists only if the noncommutative parameter θ_{ij} goes to zero at least as fast as $\hbar\rightarrow 0$, and only for $\theta_{ij}/\hbar\rightarrow 0$ as $\theta_{ij}\rightarrow 0$ this limit will lead to the usual Newtonian Mechanics. In [8] were derived the classical equations

$$\frac{d}{dt}x_i=\frac{p_i}{m}-\tilde{\theta}_{ji}\frac{\partial V}{\partial x_j}, \quad \text{and} \quad \frac{d}{dt}p_i=-\frac{\partial V}{\partial x_i}, \tag{10}$$

with the coordinates satisfying the Poisson brackets,

$$\{x_i,x_j\}=i\tilde{\theta}_{ij}, \quad \{x_i,p_j\}=\delta_{ij}, \quad \{p_i,p_j\}=0. \tag{11}$$

After introducing the variables $\tilde{x}_i=x_i+\frac{1}{2}\tilde{\theta}_{ij}p_j, \quad \tilde{p}_i=p_i$ the equations (10) can be written as

$$\begin{aligned}\frac{d}{dt}\tilde{x}_i= &\quad \frac{\tilde{p}_i}{m}-\frac{\tilde{\theta}_{ji}}{2}\frac{\partial V}{\partial\tilde{x}_j},\\ \frac{d}{dt}\tilde{p}_i= &\quad -\frac{\partial V}{\partial\tilde{x}_i}+\frac{\tilde{\theta}_{kj}}{2}\tilde{p}_j\frac{\partial^2 V}{\partial\tilde{x}_k\partial\tilde{x}_i},\end{aligned} \tag{12}$$

up to first order in $\tilde{\theta}_{ij}$. The equivalence of (9) and (12) can be proven if we consider that the classical limit of (9) exists since, in principle, we can take $\theta_{ij}=\hbar\tilde{\theta}_{ij}$ in (9). Moreover if we assume that the wave function of our system is different from zero only in a small region of space around the mean value of the coordinate, say $<\mathbf{x}>$ we can expand the derivative of the potential around the mean values $\bar{\tilde{x}}_i=<\tilde{x}_i>$. In this way we end up with relations analog to (12) for the mean values, i.e.,

$$\begin{aligned}\frac{d}{dt}\bar{\tilde{x}}_i= &\quad \frac{\bar{\tilde{p}}_i}{m}-\frac{\tilde{\theta}_{ji}}{2}\frac{\partial V}{\partial\bar{\tilde{x}}_j}\\ \frac{d}{dt}\bar{\tilde{p}}_i= &\quad -\frac{\partial V}{\partial\bar{\tilde{x}}_i}+\frac{\tilde{\theta}_{kj}}{2}\bar{\tilde{p}}_j\frac{\partial^2 V}{\partial\bar{\tilde{x}}_k\partial\bar{\tilde{x}}_i},\end{aligned} \tag{13}$$

which are valid as far as $|\frac{\partial V}{\partial\bar{\tilde{x}}_i}| \gg \frac{1}{2}|\frac{\partial^3 V}{\partial\bar{\tilde{x}}_k\bar{\tilde{x}}_j\bar{\tilde{x}}_i}| < \Delta\tilde{x}_j\Delta\tilde{x}_k>$. These inequalities are better satisfied when the potential changes smoothly with the coordinates $\tilde{x}_i$ and when the spatial extension of the wave packet is small. But as small values of $<\Delta\tilde{x}>$ lead to a large indeterminacy in the value of the momentum in order that one may approximately apply classical ideas about the motion of a particle it is necessary also to consider that $<\frac{\tilde{p}_i^2}{2\mu}>\gg\frac{<(\Delta\tilde{p}_i)^2>}{2\mu}$ holds. Thus these conditions are better satisfied when the particle is moving with a large momentum in a smoothly varying external field. And in a situation like this the center of the wave packet could move classically following a deformed Newton's Second law,

$$m\frac{d^2}{dt^2}\bar{\tilde{x}}_i=-\frac{\partial V}{\partial\bar{\tilde{x}}_i}+m\tilde{\theta}_{ij}\frac{\partial^2 V}{\partial\bar{\tilde{x}}_k\partial\bar{\tilde{x}}_j}\frac{d}{dt}\bar{\tilde{x}}_k, \tag{14}$$

which is obtained from (13) after returning to the original variables (4) and which is the Newton's Second Law in nc space obtained in [7, 8, 9], expressed now for the mean values. Note that for a single particle constrained to move in only one spatial dimension the nc term will not be present. So in principle in order to consider nc two-dimensional models one will be forced to introduce the noncommutativity of the time coordinate.

Using this deformed Newton's Second Law (14) different traditional single particle problems have been studied like the central forced potential [9], the three-dimensional harmonic oscillator [8] and the Kepler problem [13]. In the following we would like to investigate the possible formulation of some discrete models in this setup and their corresponding continuous limits such that alternative nc extensions of two-dimensional field theories will be obtained.

III. INFINITE OSCILLATORS CHAINS AND THE NEWTON'S SECOND LAW ON NC SPACE

The equation (14), as it was derived applying the Ehrenfest's theorem, will describe the dynamical evolution of a single particle with n degrees of freedom whose spatial coordinates do not commute. Our concern now is with a discrete system with $n \to \infty$ degrees of freedom but now formed by an infinite amount of interacting particles in such a way that it approximates an infinitely long rod that can undergo small longitudinal vibrations. So this system it is composed of a one-dimensional infinite chain of equal mass points interacting particles spaced a distance δ apart. But now, let us consider that the coordinates of the point particles do not commute at quantum level. In this sense we have extended the concept of noncommutativity from coordinates to degrees of freedom. Consequently if we are interested on the classical description of this system we will be dealing with Poisson brackets like (11). And in this sense the system will evolve in time following the Newton's Second law on nc space,

$$m\frac{d^2}{dt^2}x_i = -\frac{\partial V}{\partial x_i} + m\tilde{\theta}_{ij}\frac{\partial^2 V}{\partial x_k \partial x_j}\frac{d}{dt}x_k. \tag{15}$$

This consideration can be related to the fact that for example for two-dimensional field theories it is valid that that the Moyal bracket of a function and the same function spatial-translated is different from zero, i.e. $[\phi(x,t), \phi(x+a,t)]_\star \neq 0$. The description of the continuous system will be obtained as the continuous limit of the discrete system is approached. So initially let us consider elastic interactions and in this sense the corresponding Hamiltonian will be

$$H = \frac{1}{2}\sum_i \Big[m\dot{x}_i^2 + k(x_{i+1} - x_i)^2\Big], \tag{16}$$

where we have assumed that the mass points can move only along the length of the chain and we have denoted the displacement of the ith particle by x_i. Here the $\cdot$ corresponds to time-derivative. We are going to consider that only the coordinates of the nearest neighbors do not commute, so

$$\tilde{\theta}_{ij} = \begin{pmatrix} \theta & \text{for} & j = i+1, \\ -\theta & \text{for} & j = i-1, \\ 0 & \text{otherwise.} & \end{pmatrix}, \tag{17}$$

with θ a constant parameter. Hence the dynamical evolution of these coordinates governed by (15) will be

$$\begin{aligned} m\ddot{x}_i &- k(x_{i+1} - 2x_i + x_{i-1}) \\ &- m\theta k(2\dot{x}_{i+1} - 2\dot{x}_{i-1} - \dot{x}_{i+2} + \dot{x}_{i-2}) = 0. \end{aligned} \tag{18}$$

So even when we consider a potential with nearest neighbors interaction and that only the nearest neighbor coordinates don't commute we have the influence of second neighbors on the equations of motion in the form of a dissipative term.

We can find the corresponding field theory computing the continuum limit. For this purpose consider that δ approaches zero and in this case the integer index i identifying the particular mass becomes the continuous position coordinate x so instead of $x_i(t)$ we will have $\phi(x,t)$. Using the Taylor expansions for

$$\begin{aligned} x_{i\pm 1}(t) &\to \phi(x \pm \delta, t) = \phi(x,t) \pm \delta\partial_x\phi(x,t) \\ &\qquad + \ldots + O(\delta^4) \\ x_{i\pm 2}(t) &\to \phi(x \pm 2\delta, t) = \phi(x,t) \pm 2\delta\partial_x\phi(x,t) \\ &\qquad + \ldots + O(\delta^4), \end{aligned} \tag{19}$$

the equations of motion (18) become

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + 2\theta\sqrt{\mu Y}\delta^2\frac{\partial^4\phi}{\partial x^3\partial t} = 0, \tag{20}$$

after redefining the variables as $t \to t\sqrt{\frac{Y}{\mu}}$, with $\mu = \frac{m}{\delta}$ the mass per unit length and $Y = k\delta$ the Young modulus. So the wave equation has gained a correction term. In this way it contrasts with the traditional noncommutative extension of the massless free field done through the introduction of the Moyal product on the action, which due to the properties of this product the wave equation preserves its original commutative shape [16].

Considering the nc parameter as a small perturbation we can look for the solution as $\phi = \phi_0 + \theta\phi_1$ with ϕ_0 a solution of the wave equation that in general it can be expressed as $\phi_0 = f_1(x+t) + f_2(x-t)$. Introducing this expression in the equation (20) we are left with

$$\partial_{tt}^2\phi_1 - \partial_{xx}^2\phi_1 + 2\theta\delta^2\sqrt{\mu Y}\partial_{xxxx}^4(f_1 - f_2) = 0, \tag{21}$$

at first order of θ. Defining

$$F(x,t) = 2\theta\delta^2\sqrt{\mu Y}\partial_{xxxx}^4(f_1 - f_2), \tag{22}$$

the solution of (21) dependent on the initial conditions $\phi(x,0) = u(x)$, $\phi_t(x,0) = v(x)$ can be particularly expressed as

$$\begin{aligned} \phi_1(x,t) = &\quad \frac{u(x-t) + u(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} v(\xi)d\xi \\ &+ \frac{1}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} F(\xi,\tau)d\xi, \end{aligned} \tag{23}$$

for $-\infty < x < \infty$, $0 < t < \infty$ [17].

Imagine now that the infinitely long elastic rod is not elastic so we could model it as a nonlinear oscillator chain, as for example the Fermi-Pasta-Ulam model [18] where we can consider additionally quadratic forces,

$$V = \sum_i \Big[\frac{k}{2}(x_{i+1} - x_i)^2 + \frac{\alpha}{3}(x_{i+1} - x_i)^3\Big], \tag{24}$$

and it is called α-model or additional cubic forces,

$$V=\sum_i\Big[\frac{k}{2}(x_{i+1}-x_i)^2+\frac{\beta}{4}(x_{i+1}-x_i)^4\Big], \tag{25}$$

and it is called β-model. These models are closely related to many physical problems like the integrability of nonlinear equations, their soliton solutions, dynamical chaos and Bose-Einstein condensates. So, following the above procedure the corresponding equations motion of these models inside the context of a non-Newtonian Mechanics will read

$$\begin{aligned} m\ddot{x}_i = \quad & k(x_{i+1}-2x_i+x_{i-1}) \qquad \alpha-\text{model} \\ & +\alpha[(x_{i+1}-x_i)^2-(x_i-x_{i-1})^2] \\ & +m\theta\frac{d}{dt}\{k(2x_{i+1}-2x_{i-1}+x_{i-2}-x_{i+2}) \\ & +\alpha[(x_{i+1}-x_i)^2-(x_{i-1}-x_{i-2})^2 \\ & -(x_{i+2}-x_{i+1})^2+(x_i-x_{i-1})^2]\}, \qquad (26) \\ m\ddot{x}_i = \quad & k(x_{i+1}-2x_i+x_{i-1}) \qquad \beta-\text{model} \\ & +\beta[(x_{i+1}-x_i)^3-(x_i-x_{i-1})^3] \\ & +m\theta\frac{d}{dt}\{k(2x_{i+1}-2x_{i-1}+x_{i-2}-x_{i+2}) \\ & +\beta[-(x_{i-1}-x_{i-2})^3-(x_{i+2}-x_{i+1})^3 \\ & +(x_i-x_{i-1})^3+(x_{i+1}-x_i)^3]\}. \qquad (27) \end{aligned}$$

If we apply the continuum limit $\delta\to 0$, introducing the corresponding Taylor expansions (19) up to fourth order in δ we will obtain the nonlinear field theories defined by the following partial differential equations:

$$\begin{aligned} & \partial^2_{tt}\phi-[1+2\bar{\alpha}\delta\partial_x\phi]\partial^2_{xx}\phi \qquad \alpha-\text{model} \\ & -\frac{\delta^2}{12}\partial^4_{xxxx}\phi+2\theta\sqrt{\mu Y}\delta^2\frac{d}{dt}\Big[\partial^3_{xxx}\phi \\ & \qquad +2\delta\bar{\alpha}\partial_x(\partial_x\phi\partial^2_{xx}\phi)\Big]=0, \qquad (28) \\ & \partial^2_{tt}\phi-[1+3\bar{\beta}\delta^2(\partial_x\phi)^2]\partial^2_{xx}\phi \qquad \beta-\text{model} \\ & -\frac{\delta^2}{12}\partial^4_{xxxx}\phi+2\theta\sqrt{\mu Y}\delta^2\partial^4_{xxxt}\phi=0, \qquad (29) \end{aligned}$$

where we have redefined the variables $t\to t\sqrt{\frac{Y}{\mu}}$, $\bar{\alpha}=\frac{\alpha}{k}$ and $\bar{\beta}=\frac{\beta}{k}$. Note that for the β-model the deformation term is the same than for the harmonic oscillators which is a consequence of the fact that the contributions of the β-terms appear only at fifth order of δ. So the application of the deformed Newton's Second Law (15) to the dynamical evolution of the system has lead to wave equations that describe shallow water waves in classical hydrodynamics perturbed by dissipative terms in the case of the β-model and by dissipative and nonlinear terms in the case of the $\alpha-$ model (28).

A related integrable model to the previous nonlinear systems is the open Toda lattice [14]. The model consists of a one-dimensional chain of n particles with exponential nonlinear nearest neighbor interaction [14] in such a way that the potential is expressed as $V=\sum_i\left[\frac{a}{b}e^{b(x_{i+1}-x_i)}-a(x_{i+1}-x_i)\right]$. If we consider that the dynamical behavior of this system is governed by the deformed Newton's Second Law (15) the equations of motion will read:

$$\begin{aligned} m\ddot{x}_i = \quad & a\Big[e^{x_{i+1}-x_i}-e^{x_i-x_{i-1}}\Big]+m\theta ab\frac{d}{dt}\Big[e^{x_{i+1}-x_i} \\ & +e^{x_i-x_{i-1}}-e^{x_{i+2}-x_{i+1}}-e^{x_{i-1}-x_{i-2}}\Big]. \end{aligned} \tag{30}$$

From now on, by simplicity reasons, we will assume - $m=a=b=1$. And once again let us follow the previous procedure in order to compute the continuum limit of this model. For this purpose we introduce the Taylor expansions (19) up to fourth order of δ in the previous expression and after normalizing the time $t\to t\delta$ it is obtained,

$$\begin{aligned} & \partial^2_{tt}\phi-[1+\delta\partial_x\phi+\frac{\delta^2}{2}(\partial_x\phi)^2]\partial^2_{xx}\phi-\frac{\delta^2}{12}\partial^4_{xxxx}\phi \\ & \qquad +2\theta\delta^2\frac{d}{dt}\Big[\partial^3_{xxx}\phi+\delta\partial_x(\partial_x\phi\partial^2_{xx}\phi)\Big]=0 \end{aligned} \tag{31}$$

As for computing the continuum limit we have expanded the exponential it is not surprising that in this case we have obtained the same correction term than in the α model (28). But there is another way to compute the continuous limit of the Toda model, for which we rewrite the equation (30) as

$$\begin{aligned} \ddot{x}_i = e^{x_{i+1}}+e^{x_{i-1}}-2e^{x_i} \ + \ \theta\frac{d}{dt}\Big[2(e^{x_{i+1}}-e^{x_{i-1}}) \\ - \ e^{x_{i+2}}+e^{x_{i+2}}\Big], \end{aligned} \tag{32}$$

where we have considered that $x_{i+1}-x_i\to x_i$. Let us introduce $e^{x_i}=1+V_i$ in order to consider the continuous limit where $V_i(t)\to\phi(x,t)$ with the corresponding Taylor expansions (19) for $V_{i\pm1},V_{i\pm2}$ but now up to fourth order in δ. So we arrive at

$$\partial^2_{tt}u=\partial^2_{xx}u+\delta\partial^2_{xx}(u^2)+\frac{\delta^2}{12}\partial^4_{xxxx}u-2\theta\delta^2\partial^4_{xxxt}u, \tag{33}$$

with $\phi=e^{\delta u}-1$ and where the time coordinate has been redefined $t\to t\delta$. In this way we have obtained the Boussinesq equation perturbed by a nc dissipative term which is the same perturbation that appeared on the wave equation (20) and in the β-model (29). By the other side, it is well known that the Boussinesq equation leads to the KdV equation when travelling waves in only one direction are considered. We will try in the following to approximately derive from (33) the corresponding nc extension of the KdV equation. So we will obtain the qualitatively behavior of the KdV model in this nc context. For this purpose let's do the change of variables $\varphi=\frac{du}{dx}$ which after integration leads to

$$\begin{aligned} \partial^2_{tt}\varphi=\partial^2_{xx}\varphi+2\delta\partial^2_{xx}\varphi\partial_x\varphi \ + \ \frac{\delta^2}{12}\partial^4_{xxxx}\varphi \\ - \ 2\theta\delta^2\partial^4_{xxxt}\varphi+const. \end{aligned} \tag{34}$$

Let us consider that the function φ represents a wave propagating to the right, $\varphi(x,t)=\varphi(x-vt)$, where $v\neq 0$ is a parameter proportional to the velocity of propagation of the wave with the extra condition $v\neq 1$ in order not to loose terms. In this sense $\partial_x\varphi=-v\partial_t\varphi$ which we introduce on the previous equations obtaining

$$(1-\frac{1}{v^2})\partial^2_{xx}\varphi \;+\; 2\delta\partial^2_{xx}\varphi\partial_x\varphi+\frac{\delta^2}{12}\partial^4_{xxxx}\varphi \\ -\; 2\theta\frac{\delta^2}{v}\partial^4_{xxxx}\varphi+const=0. \quad (35)$$

Considering the equation for $\partial_x\varphi=\Phi$, the time derivative can be recovered introducing a parameter ϵ, $\epsilon\neq 1$ that will give the proportion of the time derivative and the spatial derivative on the terms where $\partial_x=-v\partial_t$ was introduced,

$$(1-\frac{1}{v^2})\epsilon\partial_x\Phi \;+\; (1-\frac{1}{v^2})(1-\epsilon)\partial_t\Phi+2\delta\Phi\partial_x\Phi \\ +\; \frac{\delta^2}{12}\partial^3_{xxx}\Phi-2\theta\frac{\delta^4}{v}\epsilon\partial^3_{xxx}\Phi \\ -\; 2\theta\frac{\delta^4}{v}(1-\epsilon)\partial^3_{xxt}\Phi+const=0.$$

Which after the transformation $\xi=x-\frac{\epsilon}{1-\epsilon}t$, $\tau=\frac{t}{1-\epsilon}$ and taking $const=0$ it can be cast on the form

$$\partial_\tau\Phi+a\Phi\partial_\xi\Phi+b\partial^3_{\xi\xi\xi}\Phi-\theta c\partial^3_{\xi\xi\tau}\Phi=0, \quad (36)$$

with $a=\frac{2\delta}{1-\frac{1}{v^2}}$, $b=\frac{\delta^2}{12(1-\frac{1}{v^2})}$ and $c=\frac{2\delta^3}{(v-\frac{1}{v})}$. Moreover scaling $\xi\to b^{-\frac{1}{3}}\xi$, $\tau\to-\tau$ and $\Phi\to ab^{-\frac{1}{3}\Phi}$ it can be expressed as the KdV equation in its more usual form, now with an additional nc term,

$$\partial_\tau\Phi=\Phi\partial_\xi\Phi+\partial^3_{\xi\xi\xi}\Phi+\theta\frac{2}{v}\left(\frac{12^2\delta^2}{1-\frac{1}{v^2}}\right)^{\frac{1}{3}}\partial^3_{\xi\xi\tau}\Phi. \quad (37)$$

Considering the noncommutative parameter small the additional term can be considered as a weak perturbation on the KdV equation. Note that in this case this perturbation in order to be real must also satisfy $v>1$. The effects of weak perturbations on the solutions of the KdV equation have been studied by different methods like the proposal based on the inverse scattering theory [20], or direct perturbation theory [21], or by Green function formalism [22]. So employing these methods would be possible to find the solutions and in this way to investigate the influence of the additional nc term on the soliton solutions of the original KdV equation. At this point could be interesting to compare with other nc extension of the KdV equation proposed on the literature [15, 19]. If on the proposal presented in [15, 19] we expand the Moyal-*star* product on θ we will find that the first correction term will appear for the second order of the nc parameter θ,

$$u_t+u_{xxx} \;-\; 6uu_x-\frac{\theta^2}{2}\Big[\partial^2_{xt}u\partial^3_{xtt}u \\ +\; \partial^3_{xxx}u\partial^2_{tt}u-4\partial^3_{xxt}u\partial^2_{xt}u\Big]=0. \quad (38)$$

Once again this result contrasts with the result obtained in this letter where the correction term appears at first order of θ. This fact is in some sense natural since on the nc extensions of two-dimensional integrable field theories like KdV equation, where the nc extensions are obtained starting from the original commutative field theory and introducing the Moyal product on the zero-curvature condition, the time is considered a non-commuting coordinate. By the other side in our approach time has always been a commuting coordinate, so it is expected the non correspondence of both formalisms.

The proper comparison must be done between the nc two-dimensional Moyal field theories and the field theories obtained as the continuous limits of corresponding discrete models whose dynamical evolutions will be governed by a Newton's Second Law obtained from a NCQM with a non-commuting time coordinate. In [23] was proposed a NCQM with commuting time, based on the works of Doplicher et al [24] and which it seems to preserve the unitary. Of course it would be interesting to investigate this in the future. By the moment we present these models as alternative nc formulations of field theories which emerge from the continuous limits of discrete models whose dynamical evolutions are governed by the deformed Newton's Second Law (15).

Acknowledgments

One of us(LACB) would like to thank the Perimeter Institute as well as the University of Waterloo for the kind hospitality and where the last part of this work was done. This work has been supported by CNPq-FAPEMAT.

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